Quantum II

Work Sheet

1. Consider the case where $j = 1$

a, Find the matrices representing the operators $\hat{\vec{J}}^2$, $\hat{J}_z$, $\hat{J}_\pm$, $\hat{J}_x$ and $\hat{J}_y$

b, Find the joint eigenstates of $\hat{\vec{J}}^2$ and $\hat{J}_z$ and verify that they form an orthogonal and complete basis

Sol$\underline{n}$

a, for $j=1$ the allowed values of m $-1, 0, 1$. The joint eigenstates of $\hat{\vec{J}}^2$ and $\hat{J}_z$ are $|1,-1\rangle$, $|1,0\rangle$, and $|1,1\rangle$. The matrix representations of the operators $\hat{\vec{J}}^2$ and $\hat{J}_z$ can be inferred from

$$\langle j', m' | \hat{\vec{J}}^2 | j, m\rangle = \hbar^2 j(j+1)\delta_{j',j}\delta_{m',m},$$

and

$$\langle j', m' | \hat{J}_z | j, m\rangle = \hbar m \delta_{j',j}\delta_{m',m}.$$

Therefore

$$\hat{\vec{J}}^2 = \begin{pmatrix} \langle 1,1|\hat{\vec{J}}^2|1,1\rangle & \langle 1,1|\hat{\vec{J}}^2|1,0\rangle & \langle 1,1|\hat{\vec{J}}^2|1,-1\rangle \\ \langle 1,0|\hat{\vec{J}}^2|1,1\rangle & \langle 1,0|\hat{\vec{J}}^2|1,0\rangle & \langle 1,0|\hat{\vec{J}}^2|1,-1\rangle \\ \langle 1,-1|\hat{\vec{J}}^2|1,1\rangle & \langle 1,-1|\hat{\vec{J}}^2|1,0\rangle & \langle 1,-1|\hat{\vec{J}}^2|1,-1\rangle \end{pmatrix}$$

$$= 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\hat{J}_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Similarly, using

$$\langle j', m'|\hat{J}_{\pm}|j, m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\,\delta_{j',j}\,\delta_{m', m\pm 1},$$

we can ascertain that the matrices of $\hat{J}_+$ and $\hat{J}_-$ are given by

$$\hat{J}_- = \hbar\sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{J}_+ = \hbar\sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The matrices for $\hat{J}_x$ and $\hat{J}_y$ in the $\{|j, m\rangle\}$ basis result immediately from the relations $\hat{J}_x = (\hat{J}_+ + \hat{J}_-)/2$ and $\hat{J}_y = \frac{i}{2}(\hat{J}_- - \hat{J}_+)$;

$$\hat{J}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{J}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

b, The joint eigenvectors of $\hat{J}^2$ and $\hat{J}_z$ can be obtained as follows. The matrix equation of ~~$\hat{J}_z|j,m\rangle = m\hbar|j,m\rangle$~~

$\hat{J}_z|j, m\rangle = m\hbar|j, m\rangle$ is

$$\hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = m\hbar\begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{aligned} \hbar a &= m\hbar a \\ 0 &= m\hbar b \\ -\hbar c &= m\hbar c \end{aligned}$$

The normalized solutions to these equations for $m = 1, 0, -1$ are respectively given by $a = 1$ $b = c = 0$; $a = 0$, $b = 1$, $c = 0$; and $a = b = 0$, $c = 1$; that is,

$$|1, 1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |1, 0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |1, -1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We can verify that these vectors are orthonormal

$$\langle 1, m' | 1, m \rangle = \delta_{m', m} \quad (m', m = -1, 0, 1)$$

We can also verify that they are complete:

$$\sum_{m=-1}^{1} |1, m\rangle\langle 1, m| = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}(0\ 0\ 1) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}(0\ 1\ 0) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}(1\ 0\ 0)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2, Consider again the case where $j = 1$

a, Calculate $[\hat{J}_x, \hat{J}_y]$, $[\hat{J}_x, \hat{J}_z]$

b, Verify that $\hat{J}_z^3 = \hbar^2 \hat{J}_z$ and $\hat{J}_\pm^3 = 0$

3, Find ~~the~~ the energy levels of a spin $s = \frac{3}{2}$ particle whose Hamiltonian is given by

$$\hat{H} = \frac{\alpha}{\hbar^2}(\hat{S}_x^2 + \hat{S}_y^2 - 2\hat{S}_z^2) - \frac{\beta}{\hbar}\hat{S}_z ;$$

α and β are constants.

4, Find the expression of ~~$Y_{30}(\theta, \varphi)$~~ $Y_{30}(\theta, \varphi)$ by using the relation $Y_{l0}(\theta, \varphi) = \sqrt{(2l+1)/4\pi}\, P_l(\cos\theta)$

5, Show the following commutation relations:

$$[\hat{P}_y, \hat{L}_y] = 0 \quad [\hat{P}_z, \hat{L}_x] = i\hbar \hat{P}_y, \quad [\hat{P}_z, \hat{L}_y] = -i\hbar \hat{P}_x$$

Assignment - 1
Quantum - II

1, Show that $\Delta J_x \Delta J_y = \frac{\hbar^2}{2}[j(j+1) - m^2]$, where
$\Delta J_x = \sqrt{\langle \hat{J}_x^2 \rangle - \langle \hat{J}_x \rangle^2}$ and $\Delta J_y = \sqrt{\langle \hat{J}_y^2 \rangle - \langle \hat{J}_y \rangle^2}$

2, Find the eigen-values of the spin operator $\vec{S}$ of an electron in the direction of a unit vector $\vec{n}$, assume that $\vec{n}$ lies in the XZ plane

3, Consider a system of total angular momentum $j=1$ the operator $\hat{J}_x$ is given by

$$\hat{J}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

What ~~are~~ are possible values when measuring $\hat{J}_x$?

4, Consider a system which is initially in the state

$$\psi(\theta,\varphi) = \frac{1}{\sqrt{5}} Y_{1,-1}(\theta,\varphi) + \sqrt{\frac{3}{5}} Y_{10}(\theta,\varphi) + \frac{1}{\sqrt{5}} Y_{11}(\theta,\varphi)$$

Find $\langle \psi | \hat{L}_+ | \psi \rangle$

5, Show the following commutation relations:

$[\hat{y}, \hat{L}_y] = 0$, $[\hat{y}, \hat{L}_z] = i\hbar \hat{x}$, $[\hat{y}, \hat{L}_x] = -i\hbar \hat{z}$