Chapter 1

Vectors and Matrices

Matrices

A matrix is a group of numbers (elements) that are arranged in rows and columns.

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & . & . & a_{1,n} \\ a_{2,1} & a_{2,2} & . & . & a_{2,n} \\ . & . & & & . \\ . & . & & & . \\ a_{m,1} & a_{m,2} & . & . & a_{m,n} \end{pmatrix}$$

This is said to be a matrix of order m by n.

It is convenient to abbreviation this matrix by the notation $A=(a_{ij})_{mxn}$.

This means that A is a matrix of size mxn (i.e m rows and n columns) and that we shall refer to the elements that lies at the intersection of the i^{th} *and* j^{th} *column by* a_{ij}

$A=(a_{ij})_{2x3}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$

Eg a_{12} *belongs to the first row and the second column*

Matrix Algebra

Arrange numbers in regular fashion as a matrix itself is not something terribly interesting.

The most important advantage from the kind of arrangement is that we can define matrix addition multiplication and scalar multiplication.

1. *Equality*

   Let $A=(a_{ij})_{mxn}$ *and* $B=(b_{ij})_{pxq}$ *be matrices. We say that A and B are equal (and write A=B) if the following conditions are satisfied.*

   a. $m=p$ *and* $n=q$ b. $(a_{ij})=(b_{ij})$ *for all pairs I,j*

2. *Addition*

   If $A=(a_{ij})$ *and* $B=(b_{ij})$ *and the sum of A and B is*

A+B =$(C_{ij})= (a_{ij})+(b_{ij})$

for example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$
$$B = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$
$$A + B = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 5 & 7 \end{pmatrix}$$

3. *Scalar multiplication*

   If $A=(a_{ij})$ *is matrix and k is number (scalar) the* $KA =(ka_{ij})$ *is the properties of k and A*

   E.g

$$3 \times A = 3 \times \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{pmatrix}$$

4. *Matrix Multiplication*

We can multiply two matrices to give a matrix result:

$X=AxB$

It is only possible to multiply A and B if the number of columns of A is equal to the number of rows of B. If A is an n by m matrix, and B is an m by p matrix, then X will be a n by p matrix. The definition of X is:

$$x_{i,j} = \sum_{k=1}^{m} a_{i,k} \times b_{k,j}$$

For example, we can multiply a 2 by 3 matrix and a 3 by 2 matrix, resulting in a 2 by 2 matrix:

$$\begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \end{pmatrix} \times \begin{pmatrix} 0 & 6 \\ 2 & 8 \\ 4 & 10 \end{pmatrix} = \begin{pmatrix} 1\times0+3\times2+5\times4 & 1\times6+3\times8+5\times10 \\ 7\times0+9\times2+11\times4 & 7\times6+9\times8+11\times10 \end{pmatrix} = \begin{pmatrix} 26 & 80 \\ 62 & 224 \end{pmatrix}$$

Notice what happens if we change the order of the two matrices. This time we are multiplying a 3 by 2 matrix with a 2 by 3 matrix, and the result is a 3 by 3 matrix, quite different from the previous result:

$$\begin{pmatrix} 0 & 6 \\ 2 & 8 \\ 4 & 10 \end{pmatrix} \times \begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \end{pmatrix} = \begin{pmatrix} 42 & 54 & 66 \\ 58 & 78 & 98 \\ 74 & 102 & 130 \end{pmatrix}$$

This shows that matrix multiplication is not commutative

Matrix multiplication is, however, associative and distributive. In summary:

$$A \times B \neq B \times A$$
$$(A \times B) \times C = A \times (B \times C)$$
$$(A + B) \times C = A \times C + B \times C$$

Transposition

Transposing a matrix means converting m by n matrix into an n by m matrix

It is denoted by a superscript T, eg:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$
$$A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

There is interesting relations b/n transposition and multiplication

$$(A \times B)^T = B^T \times A^T$$

Special Types of Matrix

1. *Vector*

A row vector is a matrix containing a single row,

eg

$$A = (1 \quad 2 \quad 3)$$

A column vector is a matrix containing a single row Eg

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Both of these forms can be used to represent vectors quantities

2. *Zero (Null) Matrix*

A zero, or null, matrix is one where every element is zero, eg

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

3. *Square Matrix*

A square matrix is one where the number of rows and columns are equal, eg a 2 by 2 matrix, a 3 by 3 matrix etc.

4. *Diagonal Matrix*

A diagonal matrix is a square_matrix in which all the elements are zero except for the elements on the leading diagonal, eg:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

5. *Unit Matrix*

A unit matrix is a square_matrix in which all the elements on the leading diagonal are 1, and all the other elements are 0,

eg:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

6. *Symmetric Matrix*

A symmetric matrix is a square_matrix where

$$a_{i,j} = a_{j,i}$$

for all elements. Ie, the matrix is symmetrical about the leading diagonal. For example

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$

7. *Skew Symmetric Matrix*

A skew symmetric matrix is a square matrix where

$$a_{i,j} = -a_{j,i}$$

for all elements. Ie, the matrix is anti-symmetrical about the leading diagonal. This of course requires that elements along the diagonal must be zero. For example

$$\begin{pmatrix} 0 & 2 & 3 \\ -2 & 0 & 5 \\ -3 & -5 & 0 \end{pmatrix}$$

8. *Orthogonal Matrix*

An orthogonal matrix is a square_matrix which produces a unit_matrix if it is multiplied by its own transpose. i.e:

$$A \times A^T = I$$

<u>Inverse Matrices and Determinants</u>

<u>The Inverse of a Matrix</u>

The inverse (or reciprocal) of a square matrix is denoted by the A^{-1}, and is defined by

$$A \times A^{-1} = I$$

The two matrices on the left are inverses of each other, whose product is the unit matrix. Not all matrices have an inverse, and those which don't are called singular matrices.

<u>Determinants</u>

The determinant of a square_matrix is a single number calculated by combining all the elements of the matrix. For example, the determinant of a 2 by 2 matrix is

$$\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1} \times a_{2,2} - a_{2,1} \times a_{1,2}$$

For a 3 by 3 matrix the formula is

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = a_{1,1} \times \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{2,1} \times \begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} + a_{3,1} \times \begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{vmatrix}$$

The 2 by 2 determinants are called minor. Every element in a determinant has a corresponding, formed by deleting the row and column containing that element. For determinant of order n, the minors of order (n-1).

<u>Cofactors</u>

The cofactor of an element is the minor multiplied by the appropriate sign

$$c_{i1} = (-1)^{i+1} x m_{i1}$$

Or more generally

$$c_{ij} = (-1)^{i+j} x m_{ij}$$

<u>Adjoint Matrices</u>

Every square_matrix has an adjoint matrix, found by taking the matrix of its cofactors, and transposing it, i.e if

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & . & . & a_{1,n} \\ a_{2,1} & a_{2,2} & . & . & a_{2,n} \\ . & . & & & . \\ . & . & & & . \\ a_{n,1} & a_{n,2} & . & . & a_{n,n} \end{pmatrix}$$

then the adjoint is

$$adj(A) = \begin{pmatrix} c_{1,1} & c_{2,1} & . & . & c_{n,1} \\ c_{1,2} & c_{2,2} & . & . & c_{n,2} \\ . & . & & & . \\ . & . & & & . \\ c_{1,n} & c_{2,n} & . & . & c_{n,n} \end{pmatrix}$$

Calculating the Inverse of a Matrix

After the previous slightly complex definitions, the calculation of the inverse matrix is relatively simple.

$$A^{-1} = \frac{adj(A)}{|A|}$$

Clearly, if the determinant of A is zero, the inverse cannot be calculated and the matrix is said to be singular.

In solving sets of simultaneous equation, we can express the equations in matrix form.

For example

$a_{11}x_1+a_{12}x_2+a_{13}x_3=b_1$

$a_{21}x_1+a_{22}x_2+a_{23}x_3=b_2$

$a_{31}x_1+a_{32}x_2+a_{33}x_3=b_3$

Can be written in the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\mathbf{Ax} = \mathbf{b}$$

In practice, we can solve the equations by operating on the augmented coefficient matrix i.e we write the constants terms as fourth column of the coefficient matrix to form A_b

$$\mathbf{A}_b = \begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{pmatrix}$$

which, of course, is a (3×4) matrix.

For example

$$x + y + z = 6$$
$$2x + 3y + 4z = 20$$
$$4x + 2y + 3z = 17$$

This can be written in terms of matrix form as

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 20 \\ 17 \end{pmatrix}$$

Or more generally

AX=B

To solve this we multiply both sides by the invers of A

$A^{-1}AX=A^{-1}B$

Then the answer becomes

$$\begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 3\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ 2\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 6 \\ 20 \\ 17 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

<u>Gaussian elimination method</u>

Once again we will demonstrate the method by a typical example

Example 1

2x-3y+2z=9

3x+2y-z=4

x-4y+2z=6

first write in terms of matrix as follows

$$\begin{pmatrix} 2 & -3 & 2 \\ 3 & 2 & -1 \\ 1 & -4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \\ 6 \end{pmatrix}$$

We then form the augment coefficient matrix by including the constants as an extra column on the right-hand side of the matrix

$$\left(\begin{array}{ccc:c} 2 & -3 & 2 & 9 \\ 3 & 2 & -1 & 4 \\ 1 & -4 & 2 & 6 \end{array}\right)$$

Now we operate on the rows to convert the first three columns into the an upper tringular matrix as follws

$$(1)\sim(3)\ \begin{pmatrix}1&-4&2&6\\3&2&-1&4\\2&-3&2&9\end{pmatrix}\qquad (2)\sim(3)\ \begin{pmatrix}1&-4&2&6\\2&-3&2&9\\3&2&-1&4\end{pmatrix}$$

$$\begin{matrix}(2)-2(1)\\(3)-3(1)\end{matrix}\ \begin{pmatrix}1&-4&2&6\\0&5&-2&-3\\0&14&-7&-14\end{pmatrix}\qquad \begin{matrix}(2)\div 5\\(3)\div 7\end{matrix}\ \begin{pmatrix}1&-4&2&6\\0&1&-\frac{2}{5}&-\frac{3}{5}\\0&2&-1&-2\end{pmatrix}$$

$$(3)-2(2)\ \begin{pmatrix}1&-4&2&6\\0&1&-\frac{2}{5}&-\frac{3}{5}\\0&0&-\frac{1}{5}&-\frac{4}{5}\end{pmatrix}\qquad (3)\times(-5)\ \begin{pmatrix}1&-4&2&6\\0&1&-\frac{2}{5}&-\frac{3}{5}\\0&0&1&4\end{pmatrix}$$

$$\boxed{\begin{pmatrix}1&-4&2\\0&1&-\frac{2}{5}\\0&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}6\\-\frac{3}{5}\\4\end{pmatrix}}$$

Expanding from the bottom row, working upwards

$$\begin{aligned}x_3 &= 4 & & & \therefore\ x_3 &= 4\\ x_2-\tfrac{2}{5}x_3 &= -\tfrac{3}{5} & \therefore\ x_2 &= -\tfrac{3}{5}+\tfrac{8}{5}=1 & \therefore\ x_2 &= 1\\ x_1-4x_2+2x_3 &= 6 & \therefore\ x_1-4+8 &= 6 & \therefore\ x_1 &= 2\end{aligned}$$

$$\therefore\ x_1=2;\quad x_2=1;\quad x_3=4$$

It is very useful method and entails fewer tedious steps , and can be used to solve efficienty higher-order sets equations and non-square systems.

Excersise

Exercise

1. (a). Show that A.(BxC)=B.(AxC)=C.(BxC)

   (b) show that Ax(BxC)=B(A.C)-C(A.B)

By the method of Gaussian elimination, solve the equations $\mathbf{Ax}=\mathbf{b}$,

where $\mathbf{A}=\begin{pmatrix}1&-2&-4\\2&1&-3\\1&3&2\end{pmatrix}$ **and** $\mathbf{b}=\begin{pmatrix}-3\\4\\5\end{pmatrix}$.

3. Apply the method of row transformation to solve the following sets of equation

a. $X_1-3x_2-2x_3=8$

$2x_1+2x_2+x_3=4$

b. $X_1-3x_2-2x_3=8$

$2x_1-x_2+x_3=9$

$3x_1-4x_2+2x_3=-3$ $3x_1+4x_2+3x_3=5$

Solve the following sets of equations by Gaussian elimination.

(a)
$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 4 \\ 2x_1 + x_2 + x_3 - 4x_4 &= 3 \\ 3x_1 - x_2 - 2x_3 + 2x_4 &= 6 \\ x_1 + 3x_2 - x_3 + x_4 &= 8 \end{aligned}$$

(b)
$$\begin{aligned} 2x_1 + 3x_2 - 2x_3 + 2x_4 &= 2 \\ 4x_1 + 2x_2 - 3x_3 - x_4 &= 6 \\ x_1 - x_2 + 4x_3 - 2x_4 &= 7 \\ 3x_1 + 2x_2 + x_3 - x_4 &= 5 \end{aligned}$$

5. Solve the following sets of equations using inverse method

a. $3x_1+2x_2-x_3=4$
$2x_1-x_2+2x_3=10$
$x_1-3x_2-4x_3=5$

b. $4x_1+5x_2+x_3=2$
$x_1-2x_2-3x_3=7$
$3x_1-x_2-2x_3=1$

6. $x_1 + 3x_2 - 2x_3 + x_4 = -1$

$2x_1 - 2x_2 + x_3 - 2x_4 = 1$

$x_1 + x_2 - 3x_3 + x_4 = 6$

$3x_1 - x_2 + 2x_3 - x_4 = 3$

Then find the value of x_1, x_2 , x_3 *and* x_4 *using Gaussian ellimination method*

7. $x+y+z=6$

$2x-y+2z=10$

$x-3y-4z=5$ *find x, y and z*

8. $4x+5y+z=2$

$x-2y-3z=7$

$3x-y-2z=1$ *find x, y and z*